THE CHEREDNIK KERNEL AND GENERALIZED EXPONENTS

BOGDAN ION

Abstract. We show how the knowledge of the Fourier coefficients of the Cherednik kernel leads to combinatorial formulas for generalized exponents. We recover known formulas for generalized exponents of irreducible representations parameterized by dominant roots, and obtain new formulas for the generalized exponents for irreducible representations parameterized by the dominant elements of the root lattice which are sums of two orthogonal short roots.

Introduction

Let $\mathfrak{g}$ be a complex simple Lie algebra of rank n and denote by G its adjoint group. The algebra $S(\mathfrak{g})$ of complex valued polynomial functions on $\mathfrak{g}$ becomes a graded representation for G. It is known from the work of Kostant [7] that if $\mathcal{I}$ denotes the subring of G–invariant polynomials on $\mathfrak{g}$ then $S(\mathfrak{g})$ is free as an $\mathcal{I}$–module and is generated by $\mathcal{H}$, the space of G–harmonic polynomials on $\mathfrak{g}$ (the polynomials annihilated by all G–invariant differential operators with constant complex coefficients and no constant term), or equivalently $S(\mathfrak{g}) = \mathcal{I} \otimes \mathcal{H}$. The space of harmonic polynomials thus becomes a graded, locally finite representation of G; it can equivalently be thought of as the ring of regular functions on the cone of nilpotent elements in $\mathfrak{g}$. If we denote by $\mathcal{H}^i$ its i–th graded piece, and by V_λ the irreducible representation of G with highest weight λ we can consider the graded multiplicity of V_λ in $\mathcal{H}$

$$E(V_\lambda) := \sum_{0\le i} \dim_{\mathbb{C}} \left(\mathrm{Hom}_G(V_\lambda, \mathcal{H}^i)\right) t^i$$

As a polynomial with positive integer coefficients $E(V_\lambda)$ can be written in the form

$$E(V_\lambda) = \sum_{i=1}^{v_\lambda} t^{e_i(\lambda)}$$

such that $e_1(\lambda) \le e_2(\lambda) \le \cdots \le e_{v_\lambda}(\lambda)$ and v_λ is the multiplicity of the 0–th weight space of V_λ. The positive integers $e_i(\lambda)$ were called by Kostant the generalized exponents of V_λ. The terminology is justified by the fact that the classical exponents

Department of Mathematics, University of Michigan, Ann Arbor, MI 48109.

E-mail address: bogdion@umich.edu.

of G, the numbers $e_1 \leq \cdots \leq e_n$ which appear in the factorization of the Poincaré polynomial of G

$$p_G(t) = \prod_{i=1}^{n}(1 + t^{2e_i+1})$$

coincide with the generalized exponents of the adjoint representation of G.

To further motivate the importance of generalized exponents note that by [8] and [4] the polynomials $E(V_\lambda)$ are particular examples of Kazhdan–Lusztig polynomials (for the affine Weyl group associated to the Weyl group of G) and therefore of considerable combinatorial complexity. The results of Lusztig and Hesselink describe $E(V_\lambda)$ as a t–analogue of the 0-th weight multiplicity of V_λ via a deformation of Kostant's weight multiplicity formula introduced by Lusztig.

The problem of computing the classical exponents of G was initially motivated by the problem of computing the Betti numbers of G . It turns out that the classical exponents admit another description quite different from the one alluded to above. It was observed independently by A. Shapiro (unpublished) and R. Steinberg [12] that if we denote by $h(k)$ the number of positive roots of height k in the root system associated to G then the number of times k occurs as an exponent of G is $h(k) - h(k + 1)$. This very simple procedure for computing the classical exponents was justified by Coleman [3] modulo the empirically observed fact that $2N = nh$ (N is the number of reflexions in the Weyl group of G and h is the order of a special element of the Weyl group called the Coxeter transformation) and by Kostant [6] who gave a uniform proof by studying the decomposition of $\mathfrak{g}$ into submodules for the action of a principal three dimensional subalgebra of $\mathfrak{g}$. There is also a proof of this fact directly from Macdonald's factorization of the Poincaré polynomial of the Weyl group of G [9] [5, Section 3.20].

The main goal of this paper is to explain how the above description of the classical exponents and similar descriptions of generalized exponents can be obtained by analyzing the Fourier coefficients of the Cherednik kernel, a certain continuous function on a maximal torus of G. Besides recovering the formulas for generalized exponents of irreducible representations parameterized by dominant roots, our main result, Theorem 4.5, describes combinatorially the generalized exponents for irreducible representations parameterized by dominant elements λ of the root lattice of $\mathfrak{g}$ which are sums of two orthogonal short roots.

To describe this result we need the following notation. Let λ be a dominant element of the root lattice of $\mathfrak{g}$ which can be written as a sum of two orthogonal short roots and it is not itself a root. For any γ in the same Weyl group orbit as λ let $n(\gamma)$ be the number of (unordered) pairs of positive short orthogonal roots which sum up to γ. Let $h_\lambda(k) := h'_\lambda(k) - h''_\lambda(k)$, where $h'_\lambda(k)$ is the number of

weights of V_λ which have height k and $h''_\lambda(k)$ is the number of weights γ of V_λ in the same Weyl group orbit as λ and whose height is $k+n(\gamma)$.

Theorem 1. *Let λ be a dominant element of the root lattice of $\mathfrak{g}$ which can be written as a sum of two orthogonal short roots and it is not itself a root. With this notation above, the multiplicity of V_λ in $\mathcal{H}^k$ equals $h(k)-h(k+1)$.*

Our result suggests that similar formulas for generalized exponents for other classes of irreducible representations of G are also possible if one explicitly describes the Fourier coefficients of the Cherednik kernel parametrized by all weights of the irreducible representation under consideration. A general technique of inductively computing the Fourier coefficients of the Cherednik kernel is described in Theorem 4.1. Another closely related method for computing the Fourier coefficients of the Cherednik kernel was introduced by Bazlov [1]. It is based on Cherednik operators and was succesfuly applied to compute the Fourier coefficients parametrized by roots, but this method seems to be less efficient in general because of the complexity of Cherednik operators.

1. Preliminaries

1.1. Let $\mathfrak{g}$ be a complex simple Lie algebra of rank n and denote by G its adjoint group. Let $\mathfrak{h}$ and $\mathfrak{b}$ be a Cartan subalgebra respectively a Borel subalgebra of $\mathfrak{g}$ such that $\mathfrak{h}\subset\mathfrak{b}$, fixed once and for all. The maximal torus of G corresponding to $\mathfrak{h}$ is denoted by H. We have $H=TA$ where T is a compact torus and A is a real split torus. The volume one Haar measure on T is denoted by ds.

Let $R\subset\mathfrak{h}^*$ be the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$, let R^+ be the set of roots of $\mathfrak{b}$ with respect to $\mathfrak{h}$ and denote by $R^-=-R^+$. Of course, $R=R^+\cup R^-$; the roots in R^+ are called positive and those in R^- negative. The set of positive simple roots determined by R^+ is denoted by $\{\alpha_1,\dots,\alpha_n\}$. We know that the roots in R have at most two distinct lengths. We will use the notation R_s and R_ℓ to refer respectively to the short roots and the long roots in R. If the root system is simply laced we consider all the roots to be short. The dominant element of R_s is denoted by θ_s and the dominant element of R_ℓ is denoted by θ_ℓ.

Any element α of R can be written uniquely as a sum of simple roots $\sum_{i=1}^n a_i\alpha_i$. The height of the root α is defined to be

$$\mathrm{ht}(\alpha)=\sum_{i=1}^n a_i.$$

The root of R with has the largest height is denoted by θ. By the above convention, if R is simply laced then $\theta=\theta_s$ and if R is not simply laced then $\theta=\theta_\ell$.

Denote by r the maximal number of laces in the Dynkin diagram associated to $\mathfrak{g}$. There is a canonical positive definite bilinear form $(\cdot,\cdot)$ on $\mathfrak{h}^*_{\mathbb{R}}$ (the real vector

space spanned by the roots) normalized such that $(\alpha,\alpha)=2$ for long roots and $(\alpha,\alpha)=2/r$ for short roots. For any root α define $\alpha^\vee=2\alpha/(\alpha,\alpha)$. We know from the axioms of a root system that $(\alpha,\beta^\vee)$ is an integer for any roots α and β. In fact, the only possible values for $|(\alpha,\beta^\vee)|$ are 0, 1 or 2 if the length of α does not exceed the length of β (the value 2 is attained only if $\alpha=\pm\beta$) and 0, r if the length of α is strictly larger than the length of β.

Define $\rho=\frac{1}{2}\sum_{\alpha\in R}\alpha^\vee$. With this notation the height of any root α can be written as $\mathrm{ht}(\alpha)=(\alpha,\rho)$. The root lattice Q is the integral span of the simple roots. For an element λ in Q we define its height as $\mathrm{ht}(\lambda)=(\lambda,\rho)$.

1.2. For any root α consider the reflexion of the Euclidean space $\mathfrak{h}_{\mathbb{R}}^*$ given by

$$s_\alpha(x)=x-(x,\alpha^\vee)\alpha.$$

The Weyl group W of the root system R is the subgroup of $\mathrm{GL}(\mathfrak{h}_{\mathbb{R}}^*)$ generated by the reflexions s_α, for all roots α (the simple reflexions $s_i:=s_{\alpha_i}$, $1\le i\le n$, are enough). The scalar product on $\mathfrak{h}_{\mathbb{R}}^*$ is equivariant with respect to the action of W.

We can extend the bilinear form on $\mathfrak{h}_{\mathbb{R}}^*$ to a bilinear form on the real vector space $V:=\mathfrak{h}_{\mathbb{R}}^*+\mathbb{R}\delta$ by setting $(\delta,V)=0$. The affine root system $\widetilde{R}$ is defined as

$$\widetilde{R}=\{\alpha+k\delta \mid \alpha\in R,\ k\in\mathbb{Z}\}.$$

The set of affine positive roots $\widetilde{R}^+$ consists of affine roots of the form $\alpha+k\delta$ such that k is positive if α is a positive root, and k strictly positive if α is a negative root. The affine simple roots are α_i $(1\le i\le n)$, and $\alpha_0:=\delta-\theta$.

The affine Weyl group $\widetilde{W}$ is the subgroup of $\mathrm{GL}(V)$ generated by all reflexions $s_{\alpha+k\delta}$ associated to affine roots. As above, the affine Weyl group is generated by the simple reflexions $s_0:=s_{\alpha_0}$, s_i $(1\le i\le n)$. Let us describe explicitly the action of the simple affine reflexion

$$s_0(x)=s_\theta(x)+(x,\theta)\delta.$$

The bilinear form on V is equivariant with respect to the affine Weyl group action.

2. The Cherednik kernel

2.1. For an element λ of the root lattice we denote by e^λ the corresponding character of the compact torus T. The trivial character e^0 will be also denoted by 1. Let $\mathbb{Z}[Q]$ be the $\mathbb{Z}$–algebra spanned by all such elements (the group algebra of the lattice Q). Note that the multiplication is given by $e^\lambda\cdot e^\mu=e^{\lambda+\mu}$. There is an involution of $\mathbb{Z}[Q]$ given by $\overline{e^\lambda}=e^{-\lambda}$. If we set $e^\delta=q$, for q a fixed complex number, the affine Weyl group acts naturally on $\mathbb{C}[Q]$. For example, $e^{s_0(\lambda)}=q^{(\lambda,\theta)}e^{s_\theta(\lambda)}$.

The subalgebra of $\mathbb{Z}[Q]$ consisting of W–invariant elements is denoted by $\mathbb{Z}[Q]^W$. The irreducible finite dimensional representations of G are parameterized by the

dominant elements of the root lattice. For a dominant λ we denote by χ_λ the character of the corresponding irreducible representation of G. Restricting the characters to T we will regard them as elements of $\mathbb{Z}[Q]$. A basis of $\mathbb{Z}[Q]^W$ is then given by the all the irreducible characters χ_λ of G.

For any continuous function f on the torus T, its Fourier coefficients are parameterized by Q and are given by

$$f_\lambda := \int_T f e^{-\lambda} ds.$$

The coefficient f_0 is called the constant term of f; it will be also denoted by $[f]$.

2.2. Let us consider the following function on the torus

$$\Delta = \frac{1}{|W|} \prod_{\alpha \in R} (1 - e^\alpha).$$

Modulo the normalization by the cardinal of the Weyl group (which makes the constant term $[\Delta] = 1$) this is the square absolute value of the Weyl denominator of the root system R. The scalar product on $\mathbb{Z}[Q]^W$ given by

$$\langle f, g \rangle := \int_T f \overline{g} \Delta ds$$

makes the characters χ_λ orthonormal.

Assume q and t are complex numbers of small absolute value and let

$$\nabla(q,t) = \prod_{\alpha \in R} \prod_{i \geq 0} \frac{1 - q^i e^\alpha}{1 - q^i t e^\alpha}$$

Since q and t are small the infinite product is absolutely convergent and $\nabla(q,t)$ should be seen as a continuous function on the torus T. In the special case when $t = q^k$ and k is a positive integer this function was introduced by Macdonald in [10] (see also [11]) and used to define a family of orthogonal polynomials associated to root systems and depending on the parameters q and t. Note that in this case $\nabla(q, q^k)$ is given by a finite product and no convergence problems appear; therefore q is not required to have small absolute value and it can be regarded as a parameter. The constant term of $\nabla(q,t)$ was subject to conjectures of Macdonald, later to be proved by Cherednik [2]. The function

$$\Delta(q,t) = \frac{\nabla(q,t)}{[\nabla(q,t)]}$$

is a W–invariant continuous function on the torus with constant term equal to one. It is also invariant under the transformation which sends e^λ, q and t to their inverses and therefore well defined also for q and t in a neighborhood of infinity. We can define the following non–degenerate scalar product on $\mathbb{Z}[Q]^W$

$$\langle f, g \rangle^\Delta_{q,t} := \int_T f \overline{g} \Delta(q,t) ds$$

2.3. Let us consider also the continuous function on T given by

$$K(q,t)=\prod_{\alpha\in R^+}\prod_{i\geq 0}\frac{(1-q^ie^{\alpha})(1-q^{i+1}e^{-\alpha})}{(1-tq^ie^{\alpha})(1-tq^{i+1}e^{-\alpha})}$$

Note that with the notation $e^{\delta}=q$ the above function can be written as

$$K(q,t)=\prod_{\alpha\in\widetilde{R}^+}\frac{1-e^{\alpha}}{1-te^{\alpha}}$$

For $t=q^k$ and positive integral k this function first appeared in Cherednik's work [2] on the Macdonald constant term conjecture. Unlike $\nabla(q,t)$ it is not invariant under the Weyl group. The function

$$C(q,t)=\frac{K(q,t)}{[K(q,t)]}$$

is a function on the torus with constant term equal to one, which we will call the Cherednik kernel. The following result establishes a very important property of $c_\lambda(q,t)$, the Fourier coefficients of $C(q,t)$.

Theorem 2.1. ([11, (5.1.10)]). *With the above notation, the Fourier coefficients $c_\lambda(q,t)$ of $C(q,t)$ are rational functions in q and t. Furthermore, the Cherednik kernel is invariant under the transformation which sends e^λ, q and t to their inverses.*

Consider now q and t as formal variables and define the field $\mathbb{F}:=\mathbb{Q}(q,t)$. We extend the involution on $\mathbb{Z}[Q]$ to the group algebra $\mathbb{F}[Q]$ by setting $\overline{q}=q^{-1}$ and $\overline{t}=t^{-1}$. Since $c_\lambda(q,t)$ are rational functions in q and t and therefore defined for generic q and t we can regard them as elements of $\mathbb{F}$. The invariance of the Cherednik kernel from the above Theorem can be restated as

$$c_\lambda(q^{-1},t^{-1})=c_{-\lambda}(q,t) \tag{1}$$

We can define the following non–degenerate scalar product on $\mathbb{F}[Q]$

$$\langle f,g\rangle^C_{q,t}:=\int_T f\overline{g}C(q,t)ds$$

For example $c_\lambda(q,t)=\langle 1,e^\lambda\rangle^C_{q,t}$. The scalar product has the property that

$$\langle g,f\rangle^C_{q,t}=\overline{\langle f,g\rangle}^C_{q,t}$$

It is known (see e.g. [11, (5.1.35)]) that the two scalar product coincide for all elements $f,g\in\mathbb{F}[Q]^W$

$$\langle f,g\rangle^\Delta_{q,t}=\langle f,g\rangle^C_{q,t} \tag{2}$$

We will use the notation $\langle\cdot,\cdot\rangle_{q,t}$ to refer to $\langle\cdot,\cdot\rangle^C_{q,t}$. It follows from the above relation that $\Delta(q,t)$ has also Fourier coefficients which are rational functions of q and t and therefore regarded as elements of $\mathbb{F}$.

2.4. For each simple affine root consider the following operator, called Demazure–Lusztig operator, acting on $\mathbb{F}[Q]$ as follows

$$T_i(e^\lambda) = e^{s_i(\lambda)} + (1-t)\frac{e^\lambda - e^{s_i(\lambda)}}{1-e^{-\alpha_i}}$$

The following result is due to Cherednik.

Theorem 2.2 ([2]). *The Demazure–Lusztig operators are unitary for the above scalar product on $\mathbb{F}[Q]$. This means that for any $f, g \in \mathbb{F}[Q]$ we have*

$$\langle T_i(f), T_i(g)\rangle_{q,t} = \langle f, g\rangle_{q,t}$$

The unitarity of the Demazure–Lusztig operators will be our main tool for computing some of the Fourier coefficients of the Cherednik kernel.

2.5. Consider now t as a formal variable. The graded torus character of $S(\mathfrak{g})$, the algebra of complex valued polynomial functions on $\mathfrak{g}$, is easily seen to be

$$\mathrm{ch}_{S(\mathfrak{g})}(t) = \prod_{\alpha\in R}\frac{1}{1-te^\alpha}$$

If χ_λ denotes the character of the irreducible representation of G with highest weight λ, then the graded multiplicity of V_λ inside $S(\mathfrak{g})$ can be computed as

$$\langle \mathrm{ch}_{S(\mathfrak{g})}(t), \chi_\lambda\rangle = \frac{1}{|W|}\int_T \nabla(0,t)\overline{\chi}_\lambda \tag{3}$$

As mentioned in Introduction if $\mathcal{I}$ denotes the subring of G–invariant polynomials on $\mathfrak{g}$ and $\mathcal{H}$ the space of G–harmonic polynomials on $\mathfrak{g}$ then $S(\mathfrak{g}) = \mathcal{I}\otimes\mathcal{H}$ as graded G–modules. If follows that if we want to compute $E(V_\lambda)$, the graded multiplicity of V_λ inside $\mathcal{H}$, then we would have to factor out in formula (3) the graded multiplicity of the trivial representation inside $S(\mathfrak{g})$, or equivalently the constant term of $\nabla(0,t)$. We can conclude that

$$E(V_\lambda) = \langle 1, \chi_\lambda\rangle^{\Delta}_{0,t} \tag{4}$$

By formula (2) we can thus express E_λ as a sum of weight multiplicities of V_λ times values of Fourier coefficients of the Cherednik kernel at $q = 0$. The non–symmetry of the Cherednik kernel allows various Fourier coefficients parameterized by elements in the same Weyl group orbit to behave differently and therefore to contribute differently to the above scalar product. This feature is not present for the Macdonald kernel $\Delta(q,t)$. We will return to the problem of computing the Fourier coefficients of the Cherednik kernel after some combinatorial considerations which will allow us to describe them in simple terms for elements of several Weyl group orbits .

3. The height function and the Bruhat order

3.1. For each w in W let $\ell(w)$ be the length of a reduced (i.e. shortest) decomposition of w in terms of the s_i. We have $\ell(w) = |\Pi(w)|$ where

$$\Pi(w) = \{\alpha \in R^+ \mid w(\alpha) \in R^-\}.$$

We also denote by ${}^c\Pi(w) = \{\alpha \in R^+ \mid w(\alpha) \in R^+\}$. If $w = s_{j_p} \cdots s_{j_1}$ is a reduced expression of w, then

$$\Pi(w) = \{\alpha^{(i)} \mid 1 \le i \le p\},$$

with $\alpha^{(i)} = s_{j_1} \cdots s_{j_{i-1}}(\alpha_{j_i})$.

For each element λ of Q define λ_+ to be the unique dominant element in $W\lambda$, the orbit of λ. Let $w_\lambda \in W$ be the unique minimal length element such that $w_\lambda(\lambda_+) = \lambda$.

Lemma 3.1. *With the notation above, we have*

$$\Pi(w_\lambda^{-1}) = \{\alpha \in R^+ \mid (\lambda, \alpha) < 0\}.$$

Proof. Let α be an element of $\Pi(w_\lambda^{-1})$. Then $w_\lambda^{-1}(\alpha)$ is a negative root and in consequence

$$0 \ge \left(\lambda_+, w_\lambda^{-1}(\alpha)\right) = (w_\lambda(\lambda_+), \alpha) = (\lambda, \alpha). \tag{5}$$

Let us see that above we cannot have equality. If $w_\lambda^{-1} = s_{j_p} \cdots s_{j_1}$ is a reduced expression, then

$$\alpha \in \Pi(w_\lambda^{-1}) = \{\alpha^{(i)} \mid 1 \le i \le p\},$$

with $\alpha^{(i)} = s_{j_1} \cdots s_{j_{i-1}}(\alpha_{j_i})$. Suppose that

$$0 = (\lambda, \alpha^{(i)}) = (\lambda, s_{j_1} \cdots s_{j_{i-1}}(\alpha_{j_i})) = (s_{j_{i-1}} \cdots s_{j_1}(\lambda), \alpha_{j_i})$$

then

$$s_{j_i} s_{j_{i-1}} \cdots s_{j_1}(\lambda) = s_{j_{i-1}} \cdots s_{j_1}(\lambda),$$

fact which contradicts the minimality of w_λ^{-1}.

Conversely, if the inequality $(\lambda, \alpha) < 0$ holds for a positive root α then equation (5) shows that $w_\lambda^{-1}(\alpha)$ is a negative root. $\square$

3.2. The Bruhat order is a partial order on any Coxeter group defined in way compatible with the length function. For an element w we put $w < s_i w$ if and only if $\ell(w) < \ell(s_i w)$. The transitive closure of this relation is called the Bruhat order. The terminology is motivated by the way this ordering arises for Weyl groups in connection with inclusions among closures of Bruhat cells for a corresponding semisimple algebraic group.

For the basic properties of the Bruhat order we refer to Chapter 5 in [5]. Let us list a few of them (the first two properties completely characterize the Bruhat order):

(1) For each $\alpha \in R^+$ we have $s_\alpha w < w$ if and only if α is in $\Pi(w^{-1})$;
(2) $w' < w$ if and only if w' can be obtained by omitting some factors in a fixed reduced decomposition of w ;
(3) if $w' \leq w$ then either $s_i w' \leq w$ or $s_i w' \leq s_i w$ (or both).

We can use the Bruhat order on W do define a partial order on each orbit of the Weyl group action on Q as follows.

Definition 3.2. *Let λ and μ be two elements of the root lattice which lie in the same orbit of W. By definition $\lambda < \mu$ if and only if $w_\lambda < w_\mu$.*

By the above Definition the dominant element of a W–orbit is the minimal element of that orbit with respect to the Bruhat order.

Lemma 3.3. *Let λ be an element of the root lattice such that $s_i(\lambda) \neq \lambda$ for some $1 \leq i \leq n$. Then $w_{s_i(\lambda)} = s_i w_\lambda$.*

Proof. Because $\ell(s_i w_\lambda) = \ell(w_\lambda) \pm 1$ and $\ell(s_i w_{s_i(\lambda)}) = \ell(w_{s_i(\lambda)}) \pm 1$ we have four possible situations depending on the choice of the signs in the above relations. The choice of a plus sign in both relations translates into $\alpha_i \notin \Pi(w_\lambda^{-1})$ and $\alpha_i \notin \Pi(w_{s_i \cdot \lambda}^{-1})$ which by Lemma 3.1 and our hypothesis implies that $(\alpha_i, \lambda) > 0$ and $(\alpha_i, s_i(\lambda)) > 0$ (contradiction). The same argument shows that the choice of a minus sign in both relations is impossible. Now, we can assume that $\ell(s_i w_\lambda) = \ell(w_\lambda) + 1$ and $\ell(s_i w_{s_i(\lambda)}) = \ell(w_{s_i(\lambda)}) - 1$, the other case being treated similarly. Using the minimal length properties of w_λ and $w_{s_i(\lambda)}$ we can write

$$\ell(w_\lambda) + 1 = \ell(s_i w_\lambda) \geq \ell(w_{s_i(\lambda)}) = \ell(s_i w_{s_i(\lambda)}) + 1 \geq \ell(w_\lambda) + 1$$

which shows that $\ell(s_i w_\lambda) = \ell(w_{s_i(\lambda)})$. Our conclusion now follows from the uniqueness of the element $w_{s_i(\lambda)}$. □

An immediate consequence is the following

Lemma 3.4. *Let λ be a weight such that $s_i(\lambda) \neq \lambda$ for some $1 \leq i \leq n$. Then $s_i(\lambda) > \lambda$ if and only if $(\alpha_i, \lambda) > 0$. If the equivalent conditions hold we also have*

$$\Pi(w_{s_i(\lambda)}) = \Pi(w_\lambda) \cup \{w_\lambda^{-1}(\alpha_i)\}.$$

Lemma 3.5. *For an element λ in the root lattice we have*

$$\mathrm{ht}(\lambda_+) - \mathrm{ht}(\lambda) = \sum_{\alpha \in \Pi(w_\lambda)} (\lambda_+, \alpha^\vee)$$

Moreover, the number $\mathrm{ht}(\lambda_+) - \mathrm{ht}(\lambda) - \ell(w_\lambda)$ *is a positive integer.*

Proof. Since $\mathrm{ht}(\lambda) = (\lambda, \rho) = (\lambda_+, w_\lambda^{-1}(\rho))$ we obtain that

$$\mathrm{ht}(\lambda_+) - \mathrm{ht}(\lambda) = (\lambda_+, \rho - w_\lambda^{-1}(\rho))$$

If we write

$$\rho = \frac{1}{2}\sum_{\alpha\in\Pi(w_\lambda^{-1})}\alpha^\vee + \frac{1}{2}\sum_{\alpha\in{}^c\Pi(w_\lambda^{-1})}\alpha^\vee$$

using the equalities

$$w_\lambda^{-1}\left(\Pi(w_\lambda^{-1})\right) = -\Pi(w_\lambda) \quad\text{and}\quad w_\lambda^{-1}\left({}^c\Pi(w_\lambda^{-1})\right) = {}^c\Pi(w_\lambda) \tag{6}$$

we find that

$$w_\lambda^{-1}(\rho) = -\frac{1}{2}\sum_{\alpha\in\Pi(w_\lambda)}\alpha^\vee + \frac{1}{2}\sum_{\alpha\in{}^c\Pi(w_\lambda)}\alpha^\vee$$

Our first claim then immediately follows. Regarding the second claim, note that for $\alpha\in\Pi(w_\lambda)$ we always have $(\lambda_+,\alpha^\vee)\geq 1$. Indeed, from the equality (6) we know that $\alpha = -w_\lambda^{-1}(\beta)$ with $\beta\in\Pi(w_\lambda^{-1})$ and therefore by Lemma 3.1

$$(\lambda_+,\alpha^\vee) = -(\lambda,\beta^\vee) > 0$$

In conclusion, $\mathrm{ht}(\lambda_+)-\mathrm{ht}(\lambda)-\ell(w_\lambda) = \sum_{\alpha\in\Pi(w_\lambda)}((\lambda_+,\alpha^\vee)-1)$ is a sum of positive integers and hence a positive integer. $\square$

For any element λ of the root lattice we will use the notation

$$D_\lambda = \mathrm{ht}(\lambda_+) - \mathrm{ht}(\lambda) - \ell(w_\lambda)$$

As we will see D_λ encodes a certain type of combinatorial information about λ.

If the root system is not simply laced it will be convenient to consider

$$D_\lambda(\ell) = \sum_{\alpha\in\Pi_\ell(w_\lambda)}((\lambda_+,\alpha^\vee)-1)$$

and

$$D_\lambda(s) = \sum_{\alpha\in\Pi_s(w_\lambda)}((\lambda_+,\alpha^\vee)-1)$$

where $\Pi_\ell(w_\lambda)$, respectively $\Pi_s(w_\lambda)$, is used to denote the long roots, respectively short roots, in $\Pi(w_\lambda)$.

3.3. Let us describe D_λ in a few cases. Assume that λ is a short root. Then $\lambda_+ = \theta_s$ and

$$D_\lambda = \sum_{\alpha\in\Pi(w_\lambda)}((\theta_s,\alpha^\vee)-1)$$

Since θ_s is a short root, it follows that the scalar product $(\theta_s,\alpha^\vee)$ equals 2 if $\alpha=\theta_s$ and equals 1 otherwise. Therefore D_λ takes the value 1 or 0 depending on whether θ_s is in $\Pi(w_\lambda)$ or not. But since $w_\lambda(\theta_s)=\lambda$ we obtain that θ_s is in $\Pi(w_\lambda)$ if and only if λ is a negative root. Therefore we have proved the following result.

Lemma 3.6. *If λ is a short root then $D_\lambda=0$ if λ is a positive root and $D_\lambda=1$ if λ is a negative root.*

3.4. In the case on non–simply laced root systems we can investigate D_λ for λ a long root. Denote first by $N(\theta_\ell)$ the number of unordered pairs $\{\alpha,\beta\}$ of short roots such that $\theta_\ell = \alpha+\beta$. For any other long root λ the number of unordered pairs $\{\alpha,\beta\}$ of short roots such that $\theta_\ell = \alpha+\beta$ is still $N(\theta_\ell)$ since w_λ provides a bijection between the set of such pairs.

If α and β are short roots such that $\theta_\ell = \alpha+\beta$ then $(\theta_\ell,\alpha^\vee) = 2 + (\beta,\alpha^\vee)$. We remark that $(\theta_\ell,\alpha^\vee)$ cannot be zero and then it equals r. It follows that always $(\beta,\alpha^\vee) = r-2$. The same is true for the scalar product of pairs of short roots associated in a similar way to any long root. Denote by $n(\lambda)$ the number of negative roots appearing in all unordered pairs of short roots such that $\lambda = \alpha+\beta$. The following result describes D_λ in combinatorial terms.

Lemma 3.7. *For a non–simply laced root system $D_\lambda(\ell) = 0$ if λ is a positive long root and $D_\lambda(\ell) = 1$ if λ is a negative long root. Also, $D_\lambda(s) = (r-1)n(\lambda)$.*

Proof. As before, by examining the scalar products we find that $D_\lambda(\ell) = 0$ if θ_ℓ is not in $\Pi(w_\lambda)$ and $D_\lambda(\ell) = 1$ if θ_ℓ is in $\Pi(w_\lambda)$. But since $w_\lambda(\theta_\ell) = \lambda$ this translates precisely into our first claim.

Regarding the second claim we use the fact that $(\theta_\ell,\alpha^\vee) = r$ for all $\alpha \in \Pi_s(w_\lambda)$ to write $D_\lambda(s) = (r-1)|\Pi_s(w_\lambda)|$. Therefore, it will be enough to show that $n(\lambda) = |\Pi(w_\lambda)|$. Remark first that all the unordered pairs of short root which sum up to λ are of the form $\{w_\lambda(\alpha), w_\lambda(\beta)\}$ with α and β positive short roots such that $\theta_\lambda = \alpha+\beta$. If, for example, $w_\lambda(\alpha)$ is a negative root then $\alpha \in \Pi_s(w_\lambda)$. We have shown that $n(\lambda) \le |\Pi(w_\lambda)|$. For the converse inequality note that if $\alpha \in \Pi_s(w_\lambda)$ then $(\theta_\ell,\alpha) = 1$ and hence $\theta_\ell - \alpha = -s_{\theta_\ell}(\alpha)$ is a short root and $\theta_\ell = \alpha + (\theta_\ell-\alpha)$. Then $\lambda = w_\lambda(\alpha) + w_\lambda(\theta_\ell-\alpha)$ and $w_\lambda(\alpha)$ is a negative root. In conclusion $|\Pi(w_\lambda)| \le n(\lambda)$ and our statement is proved. □

3.5. We will give a combinatorial description of D_λ for a few more Weyl group orbits. Let us describe first the orbits we wish to consider. Define

$$S := \{\gamma = \alpha+\beta \mid \alpha,\beta \in R_s\ ,\ (\alpha,\beta) = 0\}$$

The Weyl group acts on S and the number of orbits of this action is given by the number of dominant elements of S. It is useful to note that the set S is empty for the root system of type G_2 and that for all the other non–simply laced root systems the long roots belong to S. Let us consider J the set of connected components of the diagram obtained from the Dynkin diagram of R by removing the nodes corresponding to those simple roots for which $(\theta_s,\alpha_i^\vee) = 1$ and which contain at least one node associated to a short simple root of R. Note that each connected component as above is itself a Dynkin diagram and therefore we can associate its Weyl group W_j, root system R_j and highest short root $\theta_{s,j}$.

Lemma 3.8. *The dominant elements of S are $\theta_s + \theta_{s,j}$ for all j in J.*

Proof. We know (see e.g. [5]) that for any element x of $\mathfrak{h}_{\mathbb{R}}^*$ the stabilizer $\mathrm{stab}_W(x)$ is generated by the simple reflexions which fix x. Therefore the stabilizer of θ_s is the group generated by the simple reflexions s_i for which $(\theta_s, \alpha_i) = 0$ and using the notation above we obtain that

$$\mathrm{stab}_W(\theta_s) = \prod_{j \in J} W_j$$

If for a simple root α_i we have $(\theta_s, \alpha_i) = 0$ then α_i belongs to one of the root systems R_j and therefore $(\theta_{s,j}, \alpha_i) \geq 0$ for any $j \in J$. Hence, $(\theta_s + \theta_{s,j}, \alpha_i) \geq 0$ for any $j \in J$. If α_i is a simple root such that $(\theta_s, \alpha_i^\vee) = 1$, since $(\theta_{s,j}, \alpha_i^\vee) \geq -1$ we obtain again that $(\theta_s + \theta_{s,j}, \alpha_i) \geq 0$ for any $j \in J$. In conclusion, the elements $\theta_s + \theta_{s,j}$ are all dominant. To finish the proof we will show that any element γ of S is in fact conjugate to one of the $\theta_s + \theta_{s,j}$.

Fix an element $\gamma = \alpha + \beta$ of S such that $\alpha, \beta \in R_s$ and $(\alpha, \beta) = 0$. We can find a Weyl group element w such that $w(\alpha) = \theta_s$ and therefore $w(\gamma) = \theta_s + w(\beta)$. Moreover, $(\theta_s, w(\beta)) = 0$. This means in particular that $s_{w(\beta)}$ is an element of W which fixes θ_s. The element $s_{w(\beta)}$ of $\mathrm{stab}_W(\theta_s)$ being a reflexion it follows that $w(\beta)$ is a short root in one of the R_j. If we denote by $\theta_{s,j}$ the highest short root of R_j, we can find an element w' of W_j such that $w'w(\beta) = \theta_{s,j}$. Of course, since w' fixes θ_s we obtain that

$$w'w(\gamma) = \theta_s + \theta_{s,j}$$

Therefore, we have proved that each element of S is in the same orbit with one of the elements $\theta_s + \theta_{s,j}$, $j \in J$. □

3.6. We will investigate the possible values of the scalar products $(\theta_s + \theta'_s, \alpha^\vee)$ for positive roots α. We wish to study the cases which were not already accounted for. Hence we fix $j \in J$ such that $\theta_s + \theta_{s,j} \neq \theta_\ell$.

The possible values of the scalar product $(\theta_s, \alpha^\vee)$ are 0, 1 and 2 and the possible values of the scalar product $(\theta_{s,j}, \alpha^\vee)$ are 0, ± 1 and 2. Note that if one on the scalar products is 2 then the other one is necessarily 0 since α is either θ_s or $\theta_{s,j}$. Therefore $(\theta_s + \theta_{s,j}, \alpha^\vee) = 2$ only if $\alpha = \theta_s$, $\alpha = \theta_{s,j}$ or $(\theta_s, \alpha^\vee) = (\theta'_s, \alpha^\vee) = 1$. The other possible values of the scalar product are 0 and 1 since $\theta_s + \theta_{s,j}$ is dominant and α positive their scalar product has to be positive.

The most interesting situation is when we have $(\theta_s, \alpha^\vee) = (\theta_{s,j}, \alpha^\vee) = 1$. In the situation when we have two distinct root lengths α can potentially be a long root. In such a case $(\theta_s, \alpha) = (\theta_{s,j}, \alpha) = 1$ and therefore $s_{\theta_s} s_{\theta_{s,j}}(\alpha) = \alpha - r\theta_s - r\theta_{s,j}$ is a long root. The scalar product $(r\theta_s + r\theta_{s,j} - \alpha, \alpha) = 2(r-1)$. If $r = 3$ this leads to a contradiction and if $r = 2$ then we obtain that $\alpha = \theta_s + \theta_{s,j}$. Hence, α being a dominant long root it must equal θ_ℓ and hence $\theta_s + \theta_{s,j} = \theta_\ell$ (contradiction).

We have shown that if α is a positive root and $(\theta_s, \alpha^\vee) = (\theta_{s,j}, \alpha^\vee) = 1$ then α is necessarily short.

Let $A := \{\alpha \in R_s^+ \mid (\theta_s, \alpha^\vee) = (\theta_{s,j}, \alpha^\vee) = 1\}$. Denote by $\varphi = -s_{\theta_s} s_{\theta_{s,j}}$. We will show that φ is an involution of A without fixed points. Indeed, $\varphi(\alpha) = \theta_s + \theta_{s,j} - \alpha$ is a short root and $(\theta_s, \theta_s + \theta_{s,j} - \alpha) = 2/r - 1/r = 1/r$ and similarly $(\theta_{s,j}, \theta_s + \theta_{s,j} - \alpha) = 1/r$, showing that $\varphi(\alpha)$ is an element of A. Obviously, φ^2 is the identity. If α is fixed by φ then $\theta_s + \theta_{s,j} = 2\alpha$. Computing the scalar product with α we obtain $2/r = 4/r$ which is a contradiction. Therefore, the involution φ does not have fixed points.

3.7. One consequence of the above considerations is that A has an even number of elements. For our $j \in J$ (chosen such that $\theta_s + \theta_{s,j} \neq \theta_\ell$) denote by $n(j)$ the number of unordered pairs $\{\alpha, \beta\}$ of short orthogonal roots such that $\theta_s + \theta_{s,j} = \alpha + \beta$. Also, for $\lambda \in W(\theta_s + \theta_{s,j})$ denote by $n(\lambda)$ the number of negative roots appearing in all unordered pairs of short roots such that $\lambda = \alpha + \beta$.

Lemma 3.9. *With the notation above $n(j) = 1 + |A|/2$.*

Proof. If $\{\alpha, \beta\}$ is a pair of short orthogonal roots for which $\theta_s + \theta_{s,j} = \alpha + \beta$, then $(\theta_s + \theta_{s,j}, \alpha^\vee) = (\alpha + \beta, \alpha^\vee) = 2/p$. Such a root must necessarily be positive since otherwise $\mathrm{ht}(\alpha + \beta) < \mathrm{ht}(\theta_s)$. From previous considerations we know that either $\alpha \in A$, either $\alpha \in \{\theta_s, \theta_{s,j}\}$. Therefore, the pair $\{\alpha, \beta\}$ is $\{\theta_s, \theta_{s,j}\}$ or the pair $\{\alpha, \varphi(\alpha)\}$ for some $\alpha \in A$. It is easy to see that the number of such pairs is $1 + |A|/2$. □

The next result describes D_λ in combinatorial terms.

Lemma 3.10. *For an element $\lambda \in W(\theta_s + \theta_{s,j})$ as above we have*

$$D_\lambda = D_\lambda(s) = n(\lambda)$$

Proof. As we have argued before, there is no long root α such that $(\theta_s + \theta_{s,j}, \alpha^\vee) = 2$ and therefore $D_\lambda(\ell) = 0$. Furthermore,

$$D_\lambda = D_\lambda(s) = \sum_{\alpha \in \Pi_s(w_\lambda)} ((\theta_s + \theta_{s,j}, \alpha^\vee) - 1)$$

and since the scalar product $(\theta_s + \theta_{s,j}, \alpha^\vee)$ is at most 2 we obtain that D_λ is the number of $\alpha \in \Pi_s(w_\lambda)$ for which $(\theta_s + \theta_{s,j}, \alpha^\vee) = 2$. We know from Lemma 3.9 that a short positive root α such that $(\theta_s + \theta_{s,j}, \alpha^\vee) = 2$ gives rise an expression $\theta_s + \theta_{s,j} = \alpha + \beta$ with α and β short positive roots Therefore $\lambda = w_\lambda(\alpha) + w_\lambda(\beta)$ and the short root $w_\lambda(\alpha)$ is negative. We have shown that $n(\lambda) \geq D_\lambda$. For the converse inequality we argue as in the proof of Lemma 3.7. □

The following result is an immediate consequence of the combinatorial description of D_λ.

Lemma 3.11. *Let* $\lambda \in W(\theta_s+\theta_{s,j})$ *as above such that* $\mathrm{ht}(\lambda)=0$. *Then* $D_\lambda = n(j)$.

Proof. The claim is clear since if $\{\alpha,\beta\}$ is a pair of orthogonal short roots such that $\lambda=\alpha+\beta$ then because $\mathrm{ht}(\lambda)=0$ precisely one of α or β is a positive root and the other is a negative root. In conclusion $n(\lambda)=n(j)$. □

The next result will be useful later.

Lemma 3.12. *Let* $\lambda = s_\theta(\theta_s+\theta_{s,j})$. *Then* $D_\lambda = 2n(j)-1$ *is the root system* R *is simply laced and* $D_\lambda = n(j)$ *if the root system is not simply laced.*

Proof. Consider a pair $\{\alpha,\beta\}$ of short orthogonal positive roots such that $\alpha+\beta=\theta_s+\theta_{s,j}$. Then $\{s_\theta(\alpha),s_\theta(\beta)\}$ is pair of short orthogonal roots such that $s_\theta(\alpha)+s_\theta(\beta)=\lambda$ and all the pairs with this property arise in this way.

Assume first that R is simply laced. If $\{\alpha,\beta\}=\{\theta_s,\theta_{s,j}\}$ then $\{s_\theta(\alpha),s_\theta(\beta)\}=\{-\theta_s,\theta_{s,j}\}$. In all the other cases $\{s_\theta(\alpha),s_\theta(\beta)\}=\{\alpha-\theta_s,\beta-\theta_s\}$. Since θ_s is the highest root of R the number of negative roots appearing in the $n(j)$ unordered pairs of short roots which sum up to λ equals $2n(j)-1$.

If R is non–simply laced then $(\theta_s+\theta_{s,j},\theta)=1$ which forces of course $(\alpha+\beta,\theta)=1$. Because θ is dominant both (α,θ) and (β,θ) are positive integers and therefore one of them equals 0 (say, the first one) and the other equals 1. Hence $\{s_\theta(\alpha),s_\theta(\beta)\}=\{\alpha-\theta,\beta\}$. In conclusion the number of negative roots appearing in the $n(j)$ unordered pairs of short roots which sum up to λ equals $n(j)$. □

3.8. For the root system R we denote by $N(R)$ the number of positive roots in R. Similarly we denote by $N(R_s)$ the number of positive short roots in R and we use corresponding notation for the root systems R_j.

Lemma 3.13. *With the notation above, there are exactly* $N(R_s)N(R_{s,j})/n(j)$ *elements in the orbit* $W(\theta_s+\theta_{s,j})$.

Proof. For a fixed short root there are exactly $2N(R_{s,j})$ short roots orthogonal to it and with the sum in the prescribed orbit (since this is the situation for θ_s). Therefore, the total number of pairs of orthogonal short roots is $4N(R_s)N(R_{s,j})$. From all these pairs by taking their sum we obtain each element of the orbit $W(\theta_s+\theta_{s,j})$ exactly $2n(j)$ times. In conclusion the number of elements in the orbit $W(\theta_s+\theta_{s,j})$ has the predicted value. □

4. Fourier coefficients

In this section we will describe a general inductive procedure for computing the Fourier coefficients of the Cherednik kernel and then we apply it to find explicit formulas for the coefficients corresponding to elements in a few Weyl group orbits.

4.1. Let λ be an element of the root lattice and α_i a simple root. If $(\lambda, \alpha_i^\vee) = k > 0$ then

$$T_i(e^\lambda) = e^{s_i(\lambda)} + (1-t)(e^\lambda + \cdots + e^{\lambda-(k-1)\alpha_i})$$

Note that for $1 < j < k$ the element $\lambda - j\alpha_i$ is a convex combination of λ and $s_i(\lambda)$. Indeed

$$\lambda - j\alpha_i = (1 - j/k)\lambda + j/ks_i(\lambda)$$

In consequence they lie in Weyl group orbits strictly closer to the origin than the elements in $W\lambda$. The same is true if $(\lambda, \theta) = k > 0$

$$T_0(e^\lambda) = tq^k e^{s_\theta(\lambda)} + (t-1)(qe^{\lambda-\theta} + \cdots + q^{k-1}e^{\lambda-(k-1)\theta})$$

and for $1 < j < k$ the element $\lambda - j\theta$ is a convex combination of λ and $s_\theta(\lambda)$. Using the unitarity of the Demazure–Lusztig operators we obtain relations between Fourier coefficients of the Cherednik kernel.

Using the equality $\langle T_i(1), T_i(e^\lambda)\rangle_{q,t} = \langle 1, e^\lambda\rangle_{q,t}$ we obtain the following relations

$$tc_{s_i(\lambda)}(q,t) - c_\lambda(q,t) = (1-t)\left(c_{\lambda-\alpha_i}(q,t) + \cdots + c_{\lambda-(k-1)\alpha_i}(q,t)\right) \tag{7}$$

for all $1 \le i \le n$ such that $s_i(\lambda) > \lambda$. Also if $(\lambda, \theta) = k > 0$ we have

$$tq^k c_\lambda(q,t) - c_{s_\theta(\lambda)}(q,t) = (1-t)\left(q^{k-1}c_{\lambda-\theta}(q,t) + \cdots + qc_{\lambda-(k-1)\theta}(q,t)\right) \tag{8}$$

Fix a non–zero dominant element $\lambda_+ \in Q$ and consider the homogeneous system associated to the above equations. The unknowns are x_λ for all $\lambda \in W\lambda_+$ and the equations

$$\begin{aligned} tx_{s_i(\lambda)} - x_\lambda &= 0 \quad \text{if } 1 \le i \le n \text{ and } s_i(\lambda) > \lambda & (9)\\ tq^k x_\lambda - x_{s_\theta(\lambda)} &= 0 \quad \text{if } (\lambda, \theta) = k > 0 & (10) \end{aligned}$$

It is easy to see that from equation (9) we obtain that $x_\lambda = t^{-\ell(w_\lambda)}x_{\lambda_+}$. We also have $k := (\lambda_+, \theta) > 0$ and from equation (10) we get

$$tq^k x_{\lambda_+} - t^{-\ell(w_{s_\theta(\lambda)})}x_{\lambda_+} = 0$$

which implies that $x_{\lambda_+} = 0$ and therefore $x_\lambda = 0$ for all $\lambda \in W\lambda_+$.

Theorem 4.1. *The system given by equations (7) and (8) and $c_0(q,t) = 1$ has unique solution.*

Proof. We know that the system has at least one solution. Fix now a non–zero dominant $\lambda_+ \in Q$. Since the homogeneous system given by equations (9) and (10) has a unique solution it follows that the system given by equations (7) and (8) has at most one solution. Combining these two remarks it follows that for any $\lambda \in W\lambda_+$, $c_\lambda(q,t)$ is uniquely expressible in terms of $c_\mu(q,t)$'s, where μ lies in an orbit of W closer to the origin that λ_+. Using induction on the distance of λ_+ to the origin we obtain that our system has a unique solution. □

4.2. We will now apply the inductive procedure described in the proof of Theorem 4.1 to find the Fourier coefficients of the Cherednik kernel corresponding to a few Weyl group orbits. The orbit of W on Q closest to the origin is $W\theta_s$. If the root system is simply laced then $\theta_s = \theta$ the highest root of R. Let X_{θ_s} be the element of $\mathbb{F}$ for which

$$c_{\theta_s}(q,t) = t^{\mathrm{ht}(\theta_s)} X_{\theta_s} \tag{11}$$

Theorem 4.2. *For any short root λ we have*

$$c_\lambda(q,t) = t^{\mathrm{ht}(\lambda)+D_\lambda} X_{\theta_s} + (t^{\mathrm{ht}(\lambda)} - t^{\mathrm{ht}(\lambda)+D_\lambda}) \tag{12}$$

and $X_{\theta_s} = (1-t^{-1})/(1-qt^{\mathrm{ht}(\theta)})$.

Proof. We will show that the above formula is valid by induction on the order on the orbit $W\theta_s$ induced from the Bruhat order. For the minimal element of the orbit we have $D_{\theta_s} = 0$ (by Lemma 3.6) and the predicted formula coincides with (11).

Assuming that the predicted formula is true for λ we will show that it is true for $s_i(\lambda) > \lambda$. As explained in Lemma 3.4 this means that $(\lambda, \alpha_i^\vee) > 0$. In fact the possible values of the scalar product are 1 or 2 (only if $\lambda = \alpha_i$). If $(\lambda, \alpha_i^\vee) = 1$ then $\mathrm{ht}(s_i(\lambda)) = \mathrm{ht}(\lambda) - 1$ and $\ell(w_{s_i(\lambda)}) = \ell(w_\lambda) + 1$. It follows that $D_{s_i(\lambda)} = D_\lambda$ therefore using equation (7) we get

$$\begin{aligned} c_{s_i(\lambda)}(q,t) &= t^{-1} c_\lambda(q,t) \\ &= t^{\mathrm{ht}(\lambda)+D_\lambda-1} X_{\theta_s} + (t^{\mathrm{ht}(\lambda)-1} - t^{\mathrm{ht}(\lambda)+D_\lambda-1}) \\ &= t^{\mathrm{ht}(s_i(\lambda))+D_{s_i(\lambda)}} X_{\theta_s} + (t^{\mathrm{ht}(s_i(\lambda))} - t^{\mathrm{ht}(s_i(\lambda))+D_{s_i(\lambda)}}) \end{aligned}$$

If $(\lambda, \alpha_i^\vee) = 2$ then $\lambda = \alpha_i$ and $s_i(\lambda) = \alpha_i$. In consequence $D_\lambda = 0$ and $D_{s_i(\lambda)} = 1$. Again by equation (7) we get

$$\begin{aligned} c_{s_i(\lambda)}(q,t) &= t^{-1} c_\lambda(q,t) + (t^{-1} - 1) \\ &= X_{\theta_s} + (t^{-1} - 1) \\ &= t^{\mathrm{ht}(s_i(\lambda))+D_{s_i(\lambda)}} X_{\theta_s} + (t^{\mathrm{ht}(s_i(\lambda))} - t^{\mathrm{ht}(s_i(\lambda))+D_{s_i(\lambda)}}) \end{aligned}$$

We have thus shown that the formula (12) is valid for all short roots. To show that X_{θ_s} has the predicted value we use the equation (8).

Indeed, if the root system R is simply laced then $\theta = \theta_s$ and the equation (8) for $\lambda = \theta_s$ becomes

$$tq^2 c_{\theta_s}(q,t) = c_{-\theta_s}(q,t) + q(1-t)$$

and replacing $c_{\theta_s}(q,t)$ and $c_{-\theta_s}(q,t)$ with their formulas in terms of X_{θ_s} we obtain the desired result.

If the root system R is non–simply laced then $\theta = \theta_\ell$ and the equation (8) for $\lambda = \theta_s$ becomes

$$tqc_{\theta_s}(q,t) = c_{\theta_s - \theta}(q,t)$$

and again our result follows. □

4.3. We will describe next the Fourier coefficients of the Cherednik kernel corresponding to long roots in the case of a non–simply laced root systems.

Theorem 4.3. *For any long root λ we have*

$$c_\lambda(q,t) = t^{\mathrm{ht}(\lambda)+D_\lambda(\ell)} X_{\theta_s} + (t^{\mathrm{ht}(\lambda)} - t^{\mathrm{ht}(\lambda)+D_\lambda(\ell)}) \tag{13}$$

Proof. Let us denote by X_θ the element of $\mathbb{F}$ defined by $c_\theta(q,t) = t^{\mathrm{ht}(\theta)} X_\theta$. We will show inductively that for any long roots λ we have

$$c_\lambda(q,t) = t^{\mathrm{ht}(\lambda)+D_\lambda} X_\theta + (t^{\mathrm{ht}(\lambda)+D_\lambda(\ell)} - t^{\mathrm{ht}(\lambda)+D_\lambda}) X_{\theta_s} + (t^{\mathrm{ht}(\lambda)} - t^{\mathrm{ht}(\lambda)+D_\lambda(\ell)}) \tag{14}$$

The formula (14) is clearly true for θ. Assuming that the predicted formula is true for λ we will show that it is true for $s_i(\lambda) > \lambda$. For such an α_i the possible values of the scalar product $(\lambda, \alpha_i^\vee)$ are 1 (if α_i is a long root), 2 (if $\alpha_i = \lambda$) and r (if α_i is a short root). We analyze these cases separately.

If $(\lambda, \alpha_i^\vee) = 1$ then $\mathrm{ht}(s_i(\lambda)) = \mathrm{ht}(\lambda) - 1$ and $\ell(w_{s_i(\lambda)}) = \ell(w_\lambda) + 1$. It follows that $D_{s_i(\lambda)} = D_\lambda$ therefore using equation (7) we get the desired formula for $c_{s_i(\lambda)}(q,t)$.

If $(\lambda, \alpha_i^\vee) = 2$ then $\lambda = \alpha_i$ and $s_i(\lambda) = \alpha_i$. In consequence $D_\lambda(\ell) = 0$, $D_{s_i(\lambda)}(\ell) = 1$ and $D_{s_i(\lambda)}(s) = D_\lambda(s)$. Again equation (7) gives the predicted formula for $c_{s_i(\lambda)}(q,t)$.

If α_i is a short root and $(\lambda, \alpha_i^\vee) = r$ then $\lambda - \alpha_i, \lambda - (r-1)\alpha_i$ are short roots (for example $\lambda - \alpha_i = -s_\lambda(\alpha_i)$). From Lemma 3.4 we know that

$$\Pi(w_{s_i(\lambda)}) = \Pi(w_\lambda) \cup \{w_\lambda^{-1}(\alpha_i)\}$$

and since α_i is a short root we obtain

$$\Pi_\ell(w_{s_i(\lambda)}) = \Pi_\ell(w_\lambda) \ \text{ and } \ \Pi_s(w_{s_i(\lambda)}) = \Pi_s(w_\lambda) \cup \{w_\lambda^{-1}(\alpha_i)\}$$

Therefore, by Lemma 3.7 it follows that

$$D_{s_i(\lambda)}(\ell) = D_\lambda(\ell) \ \text{ and } \ D_{s_i(\lambda)}(s) = D_\lambda(s) + r - 1$$

One immediate consequence of the above equalities is that

$$\mathrm{ht}(s_i(\lambda)) + D_{s_i(\lambda)} = \mathrm{ht}(\lambda) + D_\lambda - 1$$

and that λ and $s_i(\lambda)$ are either both positive roots or both negative roots. Hence

$$D_{\lambda-\alpha_i} = D_{\lambda-(r-1)\alpha_i} = D_\lambda(\ell)$$

The equation (7) in this case becomes

$$c_{s_i(\lambda)}(q,t) = t^{-1}c_\lambda(q,t) + (t^{-1}-1)(c_{\lambda-\alpha_i}(q,t) + c_{\lambda-(r-1)\alpha_i}(q,t))$$

and the induction hypothesis together with the above equalities implies the formula (14) for $c_{s_i(\lambda)}(q,t)$.

We have thus shown that the formula (14) is valid for all long roots. To show that (13) is true it is enough to show that $X_\theta = X_{\theta_s}$. To see that this is indeed the case we use equation (8) for $\lambda = \theta$

$$tq^2 c_\theta(q,t) = c_{-\theta}(q,t) + q(1-t)$$

or equivalently

$$(q^2 t^{2\mathrm{ht}(\theta)} - t^{D_{-\theta}(s)})X_\theta = (1 - t^{D_{-\theta}(s)})X_{\theta_s} + (t^{-1}-1)(qt^{\mathrm{ht}(\theta)}+1)$$

Replacing $t^{-1}-1$ with $(qt^{\mathrm{ht}(\theta)}-1)X_{\theta_s}$ in the above formula immediately gives $X_\theta = X_{\theta_s}$ and therefore the desired result. □

Theorem 4.2 and Theorem 4.3 were also obtained by Bazlov [1, Theorem 3] using a different (but related) procedure for computing the Fourier coefficients based on the unitarity of Cherednik operators. Given the considerable complexity of Cherednik operators, their action is very hard to be analyzed in general.

4.4. We will next describe the Fourier coefficients for the Cherednik kernel for all $\lambda \in W(\theta_s + \theta_{s,j})$, where $j \in J$ is such that $\theta_s + \theta_{s,j} \neq \theta_\ell$. First, let us define X_j to be the unique element of $\mathbb{F}$ for which

$$c_{\theta_s+\theta_{s,j}}(q,t) = t^{\mathrm{ht}(\theta_s+\theta_{s,j})} X_j X_{\theta_s} \tag{15}$$

We will also need the following notation: for λ as above d_λ is defined to be 0 if $\mathrm{ht}(\lambda) \geq 0$ and to be 1 if $\mathrm{ht}(\lambda) < 0$. Also,

$$\begin{aligned} a_\lambda(t) &:= t^{\mathrm{ht}(\lambda)}(t^{d_\lambda} + t^{D_\lambda - n(j)} - t^{D_\lambda} - t^{d_\lambda + D_\lambda - n(j)}) \\ b_\lambda(t) &:= t^{\mathrm{ht}(\lambda)}(1 + t^{d_\lambda + D_\lambda - n(j)} - t^{d_\lambda} - t^{D_\lambda - n(j)}) \end{aligned}$$

Theorem 4.4. *For λ as above we have*

$$c_\lambda(q,t) = t^{\mathrm{ht}(\lambda)+D_\lambda} X_j X_{\theta_s} + a_\lambda(t) X_{\theta_s} + b_\lambda(t) \tag{16}$$

and $X_j = (1-t^{-n(j)})/(1-qt^{\mathrm{ht}(\theta)-n(j)+1})$

Proof. As before, we will show inductively that the formula (16) holds for any λ in the Weyl group orbit of $\theta_s + \theta_{s,j}$. The formula (16) is easily seen to be true for the dominant element of the orbit.

Assuming that the predicted formula is true for λ we will show that it is true for $s_i(\lambda) > \lambda$. According to Section 3.6 the possible values of the scalar product $(\lambda, \alpha_i^\vee)$ are 1 and 2. In the latter case α_i is necessarily a short root.

If $(\lambda, \alpha_i^\vee) = 1$ then $\mathrm{ht}(s_i(\lambda)) = \mathrm{ht}(\lambda) - 1$ and $\ell(w_{s_i(\lambda)}) = \ell(w_\lambda) + 1$. It follows that $D_{s_i(\lambda)} = D_\lambda$ and by using equation (7) we get the desired formula for $c_{s_i(\lambda)}(q,t)$. In fact, special care is needed if $\mathrm{ht}(\lambda) = 0$ since $d_\lambda = 0$ and $d_{s_i(\lambda)} = 1$, but Lemma 3.11 assures that everything goes smoothly.

If α_i is a short root and $(\lambda, \alpha_i^\vee) = 2$ then $\lambda - \alpha_i$ is also a short root which is orthogonal on α_i (see Section 3.6). From Lemma 3.4 we know that

$$\Pi(w_{s_i(\lambda)}) = \Pi(w_\lambda) \cup \{w_\lambda^{-1}(\alpha_i)\}$$

and therefore

$$D_{s_i(\lambda)} = D_\lambda + 1$$

The equation (7) in this case becomes

$$c_{s_i(\lambda)}(q,t) = t^{-1} c_\lambda(q,t) + (t^{-1} - 1) c_{\lambda - \alpha_i}(q,t)$$

Let us assume first that $\mathrm{ht}(\lambda) \neq 0$. Since $\lambda - \alpha_i$ is a short root it does not have height zero and moreover $D_{\lambda - \alpha_i} = d_\lambda = d_{s_i(\lambda)}$. Using this equality, the above equation gives

$$\begin{aligned}
a_{s_i(\lambda)}(t) &= t^{\mathrm{ht}(\lambda)-1}(t^{d_\lambda} + t^{D_\lambda - n(j)} - t^{D_\lambda} - t^{d_\lambda + D_\lambda - n(j)}) + (t^{-1} - 1) t^{\mathrm{ht}(\lambda) - 1 + d_\lambda} \\
&= t^{\mathrm{ht}(\lambda)-1}(t^{d_\lambda - 1} + t^{D_\lambda - n(j)} - t^{D_\lambda} - t^{d_\lambda + D_\lambda - n(j)}) \\
&= t^{\mathrm{ht}(s_i(\lambda))}(t^{d_{s_i(\lambda)}} + t^{D_{s_i(\lambda)} - n(j)} - t^{D_{s_i(\lambda)}} - t^{d_{s_i(\lambda)} + D_{s_i(\lambda)} - n(j)})
\end{aligned}$$

and

$$\begin{aligned}
b_{s_i(\lambda)}(t) &= t^{\mathrm{ht}(\lambda)-1}(1 - t^{d_\lambda})(1 - t^{D_\lambda - n(j)}) + t^{\mathrm{ht}(\lambda)-1}(1 - t^{d_\lambda})(t^{-1} - 1) \\
&= t^{\mathrm{ht}(\lambda)-1}(1 - t^{d_\lambda})(t^{-1} - t^{D_\lambda - n(j)}) \\
&= t^{\mathrm{ht}(s_i(\lambda))}(1 - t^{d_{s_i(\lambda)}})(1 - t^{D_{s_i(\lambda)} - n(j)})
\end{aligned}$$

which give the desired formula for $c_{s_i(\lambda)}(q,t)$. The case $\mathrm{ht}(\lambda) = 0$ is handled similarly with the use of Lemma 3.11. We have therefore proved the formula (16) is always valid. We are left with finding the precise value of X_j. As before, we will make use of equation (8) for $\theta_s + \theta_{s,j}$, which takes different forms depending on whether the root system R is simply laced or not. In both cases we will use the Lemma 3.12 which describes $D_{s_\theta(\theta_s + \theta_{s,j})}$.

If R is non–simply laced the equation (8) for $\lambda = \theta_s + \theta_{s,j}$ becomes

$$qtc_{\theta_s+\theta_{s,j}}(q,t) = c_{\theta_s+\theta_{s,j}-\theta}(q,t)$$

or equivalently

$$qt^{\mathrm{ht}(\theta_s+\theta_{s,j})+1}X_jX_{\theta_s} = t^{\mathrm{ht}(\theta_s+\theta_{s,j}-\theta)+n(j)}X_jX_{\theta_s} + t^{\mathrm{ht}(\theta_s+\theta_{s,j}-\theta)}(1-t^{n(j)})X_{\theta_s}$$

which gives the desired expression for X_j.

If R is simply laced the equation (8) for $\lambda = \theta_s + \theta_{s,j}$ becomes

$$q^2tc_{\theta_s+\theta_{s,j}}(q,t) = c_{-\theta_s+\theta_{s,j}}(q,t) + q(1-t)c_{\theta_{s,j}}(q,t)$$

After replacing each Fourier coefficient with its formula given by (16) and (12) and after straightforward manipulations we find the predicted expression for X_j. □

4.5. We will collect all the values of the Fourier coefficients at $q = 0$. For λ a positive root (short or long) we have

$$c_\lambda(0,t) = t^{\mathrm{ht}(\lambda)} - t^{\mathrm{ht}(\lambda)-1} \tag{17}$$

and for λ a negative root we have $c_\lambda(0,t) = 0$.

For λ an element of the orbit of $\theta_s + \theta_{s,j}(\neq \theta_\ell)$ we get

$$c_\lambda(0,t) = (t^{\mathrm{ht}(\lambda)} - t^{\mathrm{ht}(\lambda)-1}) - (t^{\mathrm{ht}(\lambda)+D_\lambda-n(j)} - t^{\mathrm{ht}(\lambda)+D_\lambda-n(j)-1}) \tag{18}$$

for λ of strictly positive height and $c_\lambda(0,t) = 0$ for λ of height smaller than zero.

Note that the fact that the Fourier coefficients $c_\lambda(0,t)$ a zero if λ has not strictly positive height follows directly from the definition of $K(0,t)$.

For λ a dominant element of the root lattice and a strictly positive integer i denote by $h'_\lambda(i)$ the number of weights (counted with multiplicities) of V_λ the irreducible representation of G with highest weight λ which have height i. For $\theta_s + \theta_{s,j}$ denote by $h''_{\theta_s+\theta_{s,j}}(i)$ the number of elements γ of the orbit of $\theta_s + \theta_{s,j}$ for which $\mathrm{ht}(\gamma) + D_\gamma - n(j) = i$. For λ a dominant root let $h_\lambda(i) := h'_\lambda(i)$; for $\theta_s + \theta_{s,j}$ let $h_{\theta_s+\theta_{s,j}}(i) = h'_{\theta_s+\theta_{s,j}}(i) - h''_{\theta_s+\theta_{s,j}}(i)$. Note that for the above cases $h_\lambda(1) \geq h_\lambda(2) \geq \cdots \geq h_\lambda(\mathrm{ht}(\lambda))$ and $h_\lambda(i) = 0$ for i strictly larger than $\mathrm{ht}(\lambda)$.

Theorem 4.5. *The generalized exponents of V_λ where λ a dominant root or $\theta_s+\theta_{s,j}$ are given by the elements of the partition dual to $\{h_\lambda(i)\}_{i\geq 1}$. Equivalently, the multiplicity of V_λ in $\mathcal{H}^i$ is $h_\lambda(i) - h_\lambda(i+1)$.*

Proof. The result is a direct consequence of the formulas (17) and (18). Indeed, if we denote by $\mathrm{wt}(\lambda)$ the set of weights of V_λ and for any $\gamma \in \mathrm{wt}(\lambda)$ we denote by

$m_{\lambda\gamma}$ the weight multiplicity of γ in V_λ, we have

$$\begin{aligned}\langle 1, \chi_\lambda\rangle_{0,t} &= \sum_{\gamma\in \mathrm{wt}(\lambda)} m_{\lambda\gamma} c_\gamma(0,t)\\ &= \sum_{1\le i\le \mathrm{ht}(\lambda)} h_\lambda(i)(t^i - t^{i-1}) + m_{\lambda 0}\\ &= \sum_{1\le i\le \mathrm{ht}(\lambda)} (h_\lambda(i) - h_\lambda(i+1))t^i\end{aligned}$$

Our result is proved. □

Note that for the representation V_θ (the adjoint representation of G) the above Theorem describes precisely the procedure for computing the classical exponents proved by Kostant [6].

4.6. For each type of root system we will describe next the results of the description of generalized exponents for $\lambda = \theta_s + \theta_{s,j}$, $j \in J$ provided by Theorem 4.5.

A_n: $|J| = 1$, $n(j) = 2$

$E(\lambda) = \sum_{1\le i<\frac{n}{2}} i(t^{2i} + t^{2n-2i}) + \sum_{1\le i<\frac{n-2}{2}} i(t^{2i+1} + t^{2n-2i-1}) + \lfloor\frac{n-2}{2}\rfloor t^n$

B_n: $|J| = 0$.

C_n, $n \ge 4$: $|J| = 1$, $n(j) = 3$

$E(\lambda) = \sum_{1\le i<\frac{n-1}{2}} i(t^{4i} + t^{4n-4i-4}) + \sum_{1\le i<\frac{n-2}{2}} i(t^{4i+2} + t^{4n-4i-6}) + \lfloor\frac{n-3}{2}\rfloor t^{2n-2}$

D_4: $|J| = 3$, $n(j) = 3$ for any $j \in J$. Also for all three $\lambda = \theta_s + \theta_{s,j}$ we have

$E(V_\lambda) = t^2 + t^4 + t^6$

D_n, $n \ge 5$: $|J| = 2$. If we denote $J = \{j_1, j_2\}$ we have $n(j_1) = n - 1$, $n(j_2) = 3$. With the notation $\lambda_1 = \theta_s + \theta_{s,j_1}$ and $\lambda_2 = \theta_s + \theta_{s,j_2}$ we have

$E(V_{\lambda_1}) = \sum_{i=1}^{n-1} t^{2i}$

For n even we have

$E(V_{\lambda_2}) = \sum_{i=1}^{n/2-2} \left(\lfloor\frac{i+1}{2}\rfloor(t^{2i} + t^{4n-2i-8}) + (\frac{n}{2} - \lfloor\frac{i+1}{2}\rfloor)(t^{2n-2i-6} + t^{2n+2i-2})\right) + \frac{n}{2}(t^{2n-6} + t^{2n-4} + t^{2n-2})$

For n odd we have

$E(V_{\lambda_2}) = \sum_{i=1}^{n-3}\lfloor\frac{i+1}{2}\rfloor(t^{2i} + t^{4n-2i-8}) + \sum_{i=\frac{n-3}{2}}^{n-3}(t^{2i+1} + t^{4n-2i-9}) + \frac{n-1}{2}t^{2n-4}$

E_6: $|J| = 1$, $n(j) = 4$

$E(V_\lambda) = t^2+t^3+t^4+t^5+2t^6+t^7+2t^8+2t^9+2t^{10}+t^{11}+2t^{12}+t^{13}+t^{14}+t^{15}+t^{16}$

E_7: $|J| = 1$, $n(j) = 5$

$E(V_\lambda) = t^2+t^4+2t^6+2t^8+3t^{10}+3t^{12}+3t^{14}+3t^{16}+3t^{18}+2t^{20}+2t^{22}+t^{24}+t^{26}$

E_8: $|J| = 1$, $n(j) = 7$

$$E(V_\lambda) = t^2 + t^6 + t^8 + t^{10} + 2t^{12} + 2t^{14} + t^{16} + 3t^{18} + 2t^{20} + 2t^{22} + 3t^{24} + 2t^{26} + 2t^{28} + 3t^{30} + t^{32} + 2t^{34} + 2t^{36} + t^{38} + t^{40} + t^{42} + t^{46}$$

F_4: $|J| = 0$.

G_2: $|J| = 0$.

It is interesting to note that whenever it is defined $n(j)$ equals some classical exponent for the root system in question. We also observe the following symmetry (recall that v_l is the multiplicity of the 0–th weight space of V_λ)

$$e_i(\lambda) + e_{v_\lambda - i}(\lambda) = \mathrm{ht}(\theta_s) + \mathrm{ht}(\theta_{s,j}) + 2$$

4.7. We close by noting that if $e_i = e_i(\theta)$ are the classical exponents, their symmetry observed by Chevalley (and proved by Kostant [6])

$$e_i + e_{n-i} = \mathrm{ht}(\theta) + 1 \tag{19}$$

can be also explained as follows. By examining $\Delta(q,t)$ we observe that the scalar product $\langle 1, \chi_\theta\rangle_{t,t} = \langle 1, \chi_\theta\rangle = 0$. On the other hand, it is easy to see that

$$\langle 1, \chi_\theta\rangle_{q,t} = \frac{(t^{e_1} + \cdots + t^{e_n}) - qt^{\mathrm{ht}(\theta)}(t^{-e_1} + \cdots + t^{-e_n})}{1 - qt^{\mathrm{ht}(\theta)}}$$

and therefore our convention $e_1 \leq \cdots \leq e_n$ immediately gives (19). For non–simply laced root systems the same argument gives a similar symmetry for the generalized exponents for V_{θ_s}. If n_s denotes the number of simple short roots in R then we have

$$e_i(\theta_s) + e_{n_s - i}(\theta_s) = \mathrm{ht}(\theta) + 1$$

However, this type of considerations do not explain the symmetry of the generalized exponents observed in the previous paragraph.

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